Finite axiomatizability for quasivarieties

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Definition. A lattice is *meet semi-distributive* if it satisfies the law

$$x \wedge y = x \wedge z \rightarrow x \wedge y = x \wedge (y \vee z).$$

Definition. For a quasivariety $\mathcal{K}$ and an algebra $\mathbf{A} \in \mathcal{K}$ the *relative congruence lattice of* $\mathbf{A}$ *with respect to* $\mathcal{K}$ is the lattice-ordered set

$$\mathrm{Con}_{\mathcal{K}}\,\mathbf{A} = \{\, \alpha \in \mathrm{Con}\,\mathbf{A} : \mathbf{A}/\alpha \in \mathcal{K} \,\}$$

with the operations $\alpha \wedge^{\mathcal{K}} \beta = \alpha \cap \beta$ and $\alpha \vee^{\mathcal{K}} \beta = (\alpha \vee \beta)'$, where $'$ is the *extension map* from $\mathrm{Con}\,\mathbf{A}$ to $\mathrm{Con}_{\mathcal{K}}\,\mathbf{A}$ defined as

$$\alpha' = \bigcap \{\, \gamma \in \mathrm{Con}_{\mathcal{K}}\,\mathbf{A} : \alpha \leq \gamma \,\}.$$

Definition. A quasivariety $\mathcal{K}$ has the *extension property* if for all algebras $\mathbf{A} \in \mathcal{K}$ the extension map is a lattice homomorphism, and has the *weak extension property* if for all $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$

$$\alpha \wedge \beta = 0_{\mathbf{A}} \rightarrow \alpha' \wedge \beta' = 0_{\mathbf{A}}.$$

Theorem (K. Baker, 1977). *Every finitely generated congruence distributive variety is finitely axiomatizable.*

Theorem (R. McKenzie, 1987). *Every finitely generated residually small congruence modular variety is finitely axiomatizable.*

Theorem (R. Willard, 2000). *Every congruence meet semi-distributive variety with a finite residual bound is finitely axiomatizable.*

Theorem (D. Pigozzi, 1988). *Every finitely generated relatively congruence distributive quasivariety is finitely axiomatizable.*

Conjecture (R. E. Park, 1976). *Every variety with a finite residual bound is finitely axiomatizable.*

Conjecture (D. Pigozzi). *Every finitely generated relatively modular quasivariety is finitely axiomatizable.*

Definition. A lattice is *pseudo-complemented* if for every element x there exists a largest element y such that $x \wedge y = 0$. In an algebraic lattice this is equivalent to $x \wedge y = x \wedge z = 0 \to x \wedge (y \vee z) = 0$.

Definition. A *set of Willard terms* for a quasivariety $\mathcal{K}$ is a finite sequence $\{ (f_i, g_i) : i < n \}$ of pairs of ternary terms such that the equations $f_i(x, y, x) \approx g_i(x, y, x)$ $(i < n)$ hold in $\mathcal{K}$ and so does

$$x \neq y \to \bigvee_{i<n} \Big(f_i(x, x, y) = g_i(x, x, y) \leftrightarrow f_i(x, y, y) \neq g_i(x, y, y) \Big).$$

Theorem. *For every quasivariety*

$$\mathrm{CD} \Rightarrow \mathrm{SD}(\wedge) \Rightarrow \mathrm{PCC} \Leftrightarrow \mathrm{W}$$

$$\mathcal{K}\text{-CD} \Rightarrow \mathcal{K}\text{-SD}(\wedge) \Leftrightarrow \mathcal{K}\text{-PCC} \Rightarrow \mathrm{W}$$

$$\mathcal{K}\text{-CD} \Rightarrow \mathrm{EP} \Rightarrow \mathrm{WEP}$$

Theorem. *A locally finite quasivariety $\mathcal{K}$ has pseudo-complemented congruences iff no algebra in $\mathcal{K}$ has a non-trivial Abelian congruence.*

Definition. An element p of a lattice is *pseudo-prime* if $x \wedge y \neq 0$ whenever $x, y \nleq p$.

Theorem. *An algebraic lattice* $\mathbf{L}$ *is pseudo-complemented iff the meet of all pseudo-prime elements is zero.*

Proof. Take a compact element $c \neq 0$, and a maximal filter F containing c but not 0. If $u \notin F$ then there exists $v \in F$ such that $u \wedge v = 0$. Since $\mathbf{L}$ is pseudo-complemented, $L - F$ is an ideal. Put $p = \bigvee(L - F)$. Note that the compact elements below p are in $L - F$, in particular $c \nleq p$, but p might be in F. Now if $x, y \nleq p$, then $x, y \in F$ and therefore $x \wedge y \neq 0$. □

Theorem. *In a pseudo-complemented algebraic lattice with countable many compact elements the meet of all meet-prime elements is zero.*

Theorem. *There exists a pseudo-complemented algebraic lattice with* $\aleph_1$ *many compact elements such that the meet of all meet-prime elements is non-zero.*

Take an algebra $\mathbf{A}$ with a pseudo-complemented congruence lattice, and $a, b \in A$, $a \neq b$. Then there exists a pseudo-prime congruence $\vartheta \in \operatorname{Con}\mathbf{A}$ such that $(a,b) \notin \vartheta$. Consider the algebra $\mathbf{A}/\vartheta$ and suppose that $\mathbf{A}/\vartheta$ does not satisfy some universal sentence

$$(\forall \bar{x}) \bigwedge_{i} \left(\bigvee_{j<n_i} s_{ij}(\bar{x}) \approx t_{ij}(\bar{x}) \vee \bigvee_{k<m_i} s'_{ik}(\bar{x}) \not\approx t'_{ik}(\bar{x}) \right).$$

So there exists a tuple $\bar{a}$ in A and i such that

$$\mathrm{Cg}_{\mathbf{A}}(s_{ij}(\bar{a}), t_{ij}(\bar{a})) \not\leq \vartheta \quad \text{and} \quad \mathrm{Cg}_{\mathbf{A}}(s'_{ik}(\bar{a}), t'_{ik}(\bar{a})) \leq \vartheta$$

for all $j < n_i$ and $k < m_i$. From this it follows that there exists a pair of elements (u, v) such that

$$(u,v) \in \mathrm{Cg}_{\mathbf{A}}(s_{ij}(\bar{a}), t_{ij}(\bar{a})) \quad \text{and} \quad \mathrm{Cg}_{\mathbf{A}}(u,v) \cap \mathrm{Cg}_{\mathbf{A}}(s'_{ik}(\bar{a}), t'_{ik}(\bar{a})) = 0_{\mathbf{A}}$$

for all $j < n_i$ and $k < m_i$.

Definition. For an algebra $\mathbf{A}$ and integer m the *principal congruence m-disjointness relation over* $\mathbf{A}$ is the $2m$-ary relation

$$PCD_m(\bar{a};\bar{b}) \stackrel{\text{def}}{=} \bigcap_{i<m} \mathrm{Cg}_{\mathbf{A}}(a_i, b_i) = 0_{\mathbf{A}}.$$

Theorem. *Let $\mathcal{W}$ be a quasivariety with pseudo-complemented congruence lattices and ϕ be a universal sentence. There exists a congruence condition $PCD(\phi)$ using the PCD_m relation such that*

$$\{\, \mathbf{A} \in \mathcal{W} : \mathbf{A} \models PCD(\phi) \,\} \subseteq \mathcal{W} \cap SP(\mathrm{Mod}(\phi)). \qquad (\star)$$

Theorem. *If ϕ is a positive universal sentence then in $(\star)$ the two classes are equal.*

Theorem. *If there is a quasivariety $\mathcal{E}$ with the weak extension property such that $W \cap SP(\mathrm{Mod}(\phi)) \subseteq \mathcal{E} \subseteq SP(\mathrm{Mod}(\phi))$ then in $(\star)$ the two classes are equal.*

Definition. For a class $\mathcal{K}$ of algebras and an integer n we define $\mathcal{K}_n$ to be the class of algabras having at most n elements.

Theorem. *Let $\mathcal{W}$ be a quasivariety with Willard terms, and n, m be integers. Then PCD_m is first-order definable in $\mathcal{W} \cap SP(H(\mathcal{W})_n)$ by a formula pcd_m. Moreover, in all algebras of this signature*

$$(\forall \bar{x}, \bar{y})(PCD_m(x;y) \to pcd_m(x;y)).$$

A version of the previous theorem was discovered independently by K. Baker, G. McNulty and Ju. Wang for congruence meet semi-distributive varieties.

Theorem. *There exists a first-order sentence $\gamma(pcd_m)$ such that for any algebra $\mathbf{A}$ with a pseudo-complemented congruence lattice,*

$$\mathbf{A} \models \gamma(pcd_m) \quad \textit{iff} \quad \mathbf{A} \models (\forall \bar{x}, \bar{y})(PCD_m(x;y) \leftrightarrow pcd_m(x;y)).$$

Theorem. *Let $\mathcal{W}$ be a quasivariety with pseudo-complemented congruence lattices, n be a positive integer, and ϕ be a positive universal sentence. Then $\mathcal{L} = \mathcal{W} \cap SP(H(\mathcal{W})_n) \cap SP(\mathrm{Mod}(\phi))$ is finitely axiomatizable relative to $\mathcal{W}$.*

Corollary. *Every quasivariety having pseudo-complemented congruence lattices and contained in a finitely generated quasivariety, is contained in a finitely axiomatizable, locally finite quasivariety.*

Corollary. *Let $\mathcal{K}$ be a finite set of finite algebras such that $SP(\mathcal{K})$ has pseudo-complemented congruence lattices. If $HS(\mathcal{K}) \subseteq SP(\mathcal{K})$ then $SP(\mathcal{K})$ is finitely axiomatizable.*

Theorem. *Let $\mathcal{W}$ be a quasivariety with pseudo-complemented congruence lattices, n be a positive integer, and ϕ be a universal sentence. Then $\mathcal{L} = \mathcal{W} \cap SP(H(\mathcal{W})_n) \cap SP(\mathrm{Mod}(\phi))$ is finitely axiomatizable relative to $\mathcal{W}$, provided there exists some quasivariety $\mathcal{E}$ with the weak extension property such that $\mathcal{L} \subseteq \mathcal{E} \subseteq SP(\mathrm{Mod}(\phi))$.*

Corollary. *Every finitely generated quasivariety with pseudo-complemented congruence lattices and the weak extension property is finitely axiomatizable.*